

TORUS INVARIANT DIVISORS

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ABSTRACT. Using the language of [AH06], and [AHS08] we describe invariant divisors on normal varieties X which admit an effective codimension one torus action. In this picture X is given by a divisorial fan on a smooth projective curve Y . Cartier divisors on X can be described by piecewise affine functions h on the divisorial fan \mathcal{S} whereas Weil divisors correspond to certain zero and one dimensional faces of it. Furthermore we provide descriptions of the divisor class group and the canonical divisor. Global sections of line bundles $\mathcal{O}(D_h)$ will be determined by a subset of a weight polytope associated to h , and global sections of specific line bundles on the underlying curve Y .

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1. INTRODUCTION

Although toric geometry covers only a rather restricted class of varieties, it nevertheless provides a large amount of toy models and fruitful examples. In order to extend its results and techniques to a broader class of objects we suggest to consider varieties admitting an effective action of a lower dimensional torus, so called T -varieties.

In particular one can consider T -varieties of codimension one, i.e. normal varieties X of dimension n which admit an effective action of the torus T^{n-1} . This setting is in some way closest to the toric one, and there have already been several approaches, e.g. in [KKMB73, IV, §1] via toroidal embeddings, in [Tim97] via the language of hypercones and hyperfans. The easiest class of examples is given by \mathbb{C}^* -surfaces, which have been studied in great detail, cf. e.g. [OW77], and [FZ03] and references therein.

However, this article will provide an insight into T -invariant divisors on X , viz. Cartier and Weil divisors using the rather new language of polyhedral divisors. For comparison, the reader may consult [KKMB73, II, §§1,2] and [Tim00].

In section 2 we recall the language of T -varieties from [AH06], and [AHS08]. As we will specialize to codimension one actions we display the essential features of this case. For such a T -variety X the building blocks consist of a smooth projective curve Y as an algebro-geometric, and an $(n - 1)$ -dimensional divisorial fan \mathcal{S} on Y as a combinatorial datum.

Section 3 deals with invariant divisors. Firstly we consider Cartier divisors. Like in toric geometry they will be related to piecewise affine linear functions h on the divisorial fan \mathcal{S} . Secondly comes the description of Weil divisors which will follow easily from the orbit structure of X lying over Y . We also include a formula for the divisor class group, and a representation of the canonical divisor. From this we then obtain a description of the global sections of a line bundle $\mathcal{O}(D_h)$ via a weight polytope \square_h associated to h , and global sections of specific line bundles on Y induced by elements of \square_h .

Section 4 completes this paper by comparing our results with those of [FZ03] in the case of affine \mathbb{C}^* -surfaces.

2. T-VARIETIES

We follow the notation of [AHS08]. First let us recall some facts and notations from convex geometry. Let N denote a lattice and $M := \text{Hom}(N, \mathbb{Z})$ its dual. The associated \mathbb{Q} -vector spaces $N \otimes \mathbb{Q}$ and $M \otimes \mathbb{Q}$ are denoted by $N_{\mathbb{Q}}$, and $M_{\mathbb{Q}}$, respectively. Let $\sigma \subset N_{\mathbb{Q}}$ be a pointed convex polyhedral cone. Consider a polyhedron Δ which can be written as a Minkowski sum $\Delta = \pi + \sigma$ of σ , and a compact polyhedron π . Then Δ is said to have σ as its *tail cone*. This decomposition of Δ is only unique up to π .

With respect to Minkowski addition the polyhedra with tail cone σ form a semi-group which we denote by $\text{Pol}_{\sigma}^{+}(N)$. Note that $\sigma \in \text{Pol}_{\sigma}^{+}(N)$ is the neutral element of this semi-group and that \emptyset by definition is also an element of $\text{Pol}_{\sigma}^{+}(N)$.

Definition 2.1. A polyhedral divisor with tail cone σ on a normal variety Y is a formal finite sum

$$\mathcal{D} = \sum_Z \Delta_Z \otimes Z,$$

where Z runs over all prime divisors on Y and $\Delta_Z \in \text{Pol}_{\sigma}^{+}(N)$. Here, finite means that only finitely many coefficients differ from the tail cone.

For every element $u \in \sigma^{\vee} \cap M$ we can consider the evaluation of \mathcal{D} via

$$\mathcal{D}(u) := \sum_Z \min_{v \in \Delta_Z} \langle u, v \rangle Z.$$

This yields an ordinary divisor on $\text{Loc } \mathcal{D}$, where

$$\text{Loc } \mathcal{D} := Y \setminus \left(\bigcup_{\Delta_Z = \emptyset} Z \right)$$

denotes the *locus* of \mathcal{D} .

Definition 2.2. A polyhedral divisor \mathcal{D} is called *proper* if

- (1) it is *Cartier*, i.e. $\mathcal{D}(u)$ is Cartier for every $u \in \sigma^{\vee} \cap M$,
- (2) it is *semiample*, i.e. $\mathcal{D}(u)$ is semiample for every $u \in \sigma^{\vee} \cap M$,
- (3) \mathcal{D} is *big* outside the boundary, i.e. $\mathcal{D}(u)$ is big for every u in the relative interior of σ^{\vee} .

From now on we will only say polyhedral divisor instead of proper polyhedral divisor except in the cases where we want to distinguish between them explicitly.

One can associate an M -graded k -algebra with such a polyhedral divisor, and consequently an affine scheme admitting a T^N -action:

$$X(\mathcal{D}) := \operatorname{Spec} \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\mathcal{O}(\mathcal{D}(u))).$$

We know that this construction gives an affine normal variety of dimension $\dim Y + \dim N$ together with a T^N -action. Moreover, every normal affine variety with torus action can be obtained this way [AH06].

Definition 2.3. Let $\mathcal{D} = \sum_Z \Delta_Z \otimes Z$, and $\mathcal{D}' = \sum_Z \Delta'_Z \otimes Z$ be two polyhedral divisors on Y .

- (1) We write $\mathcal{D}' \subset \mathcal{D}$ if $\Delta'_Z \subset \Delta_Z$ holds for every prime divisor Z .
- (2) Let $\sigma := \operatorname{tail} \mathcal{D}$ denote the tailcone of \mathcal{D} . For an element $u \in \sigma^\vee \cap M$ we define $\operatorname{face}(\sigma, u)$ to be the set of all $v \in \sigma$ such that $\langle u, v \rangle$ is minimal.
- (3) We define the *intersection* of polyhedral divisors by

$$\mathcal{D} \cap \mathcal{D}' := \sum_Z (\Delta'_Z \cap \Delta_Z) \otimes Z.$$

- (4) We define the *degree* of a polyhedral divisor \mathcal{D} on a curve Y as

$$\deg \mathcal{D} := \sum_Z \Delta_Z.$$

Note: If \mathcal{D} carries \emptyset -coefficients we get $\deg \mathcal{D} = \emptyset$.

- (5) For a (not necessarily closed) point $y \in Y$ we define the *fiber polyhedron*

$$\mathcal{D}_y := \sum_{y \in Z} \Delta_Z.$$

Note: We can recover Δ_Z this way since $\Delta_Z = \mathcal{D}_Z$.

- (6) For an open subset $U \subset Y$ we set

$$\mathcal{D}|_U := \mathcal{D} + \sum_{Z \cap U = \emptyset} \emptyset \otimes Z.$$

Now assume that $\mathcal{D}' \subset \mathcal{D}$ holds and $\mathcal{D}, \mathcal{D}'$ are proper. This implies

$$\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\mathcal{O}(\mathcal{D}'(u))) \supset \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\mathcal{O}(\mathcal{D}(u))),$$

and we get a dominant morphism $X(\mathcal{D}') \rightarrow X(\mathcal{D})$.

Definition 2.4. If $\mathcal{D}' \subset \mathcal{D}$ holds for two proper polyhedral divisors and the corresponding map defines an open inclusion, then we say that \mathcal{D}' is a *face* of \mathcal{D} , and we denote this by $\mathcal{D}' \prec \mathcal{D}$.

Definition 2.5.

- (1) A *divisorial fan* is a finite set \mathcal{S} of proper polyhedral divisors such that for $\mathcal{D}, \mathcal{D}' \in \mathcal{S}$ we have $\mathcal{D} \succ \mathcal{D}' \cap \mathcal{D} \prec \mathcal{D}'$.
- (2) The polyhedral complex \mathcal{S}_y defined by the polyhedra \mathcal{D}_y is called a *slice* of the divisorial fan \mathcal{S} .
- (3) \mathcal{S} is called *complete* if all slices \mathcal{S}_y are complete subdivisions of $N_{\mathbb{Q}}$.

The upper face relations guarantee that we can glue the affine varieties $X(\mathcal{D})$ via

$$X(\mathcal{D}) \leftarrow X(\mathcal{D} \cap \mathcal{D}') \rightarrow X(\mathcal{D}').$$

By [AHS08, 5.4.] we know that the cocycle condition is fulfilled, so we obtain a variety which we denote by $X(\mathcal{S})$. In the case of a complete this variety is also complete.

A divisorial fan \mathcal{S} corresponds to an open affine covering of $X(\mathcal{S})$ given by $(X(\mathcal{D}))_{\mathcal{D} \in \mathcal{S}}$. Observe that it is not unique, because we may switch to another invariant open affine covering of the same variety. We will do this occasionally by refining an existing divisorial fan.

Let us consider the affine case $X = \text{Spec } A$, where $A = \bigoplus_u \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}(\mathcal{D}(u)))$. We have $A_0 = \Gamma(\text{Loc } \mathcal{D}, \mathcal{O}_{\text{Loc } \mathcal{D}})$, and thus get the following two proper and surjective maps to $Y_0 := \text{Spec } A_0$, the categorical quotient of X :

$$q : X \rightarrow Y_0, \quad \pi : \text{Loc } \mathcal{D} \rightarrow Y_0.$$

Lemma 2.6. *Let \mathcal{D} be a polyhedral divisor on Y and $\{U_i\}_{i \in I}$ an open affine covering of Y_0 . Then $q^{-1}(U_i) \cong X(\mathcal{D}|_{\pi^{-1}(U_i)})$. Moreover we get a divisorial fan $\mathcal{S} := \{\mathcal{D}|_{\pi^{-1}(U_i)}\}_{i \in I}$ such that $X(\mathcal{S}) \cong X(\mathcal{D})$.*

Proof. This is a direct consequence of [AHS08, 3.3]. \square

Remark 2.7. We pay special attention to the case that a torus of dimension $\dim X - 1$ acts on X . This means that the underlying variety Y of the corresponding divisorial fan is a projective curve.

In this case the locus of a polyhedral divisor may be affine or complete, and we get simple criteria for properness and the face relations:

- \mathcal{D} is a proper polyhedral divisor if $\deg \mathcal{D}$ is strictly contained in $\text{tail } \mathcal{D}$ and for every $u \in \sigma^\vee$ with

$$\text{face}(\text{tail } \mathcal{D}, u) \cap \deg \mathcal{D} \neq \emptyset$$

some multiple of $\mathcal{D}(u)$ is principal.

- Given two polyhedral divisors $\mathcal{D}' \subset \mathcal{D}$ with \mathcal{D} being proper, then \mathcal{D}' is proper and a face of \mathcal{D} if and only if Δ'_P is a face of Δ_P for every point $P \in Y$ and we have $\deg \mathcal{D} \cap \text{tail } \mathcal{D}' = \deg \mathcal{D}'$.

Observe that $X(\mathcal{S})$ is not determined by the prime divisor slices \mathcal{S}_D of \mathcal{S} in general. This can already be seen in the case of toric surfaces with restricted torus action (cf. 2.9). Considering the Hirzebruch surface \mathbb{F}_1 we could blow up the point corresponding to the cone σ_2 , thus inserting the ray $\mathbb{R}_{\geq 0}(-1, 0)$, and nevertheless obtain the same slices. So merely looking at the subdivisions, i.e. the slices \mathcal{S}_D , does not give us all the necessary information. We really need to know which polyhedra in different slices belong to the same polyhedral divisor.

For a divisorial fan on a curve which consists only of polyhedral divisors with affine locus the situation is different. If we consider two such fans $\mathcal{S}, \mathcal{S}'$ having the same slices, 2.6 tells us that there exists a common refinement

$$\mathcal{S}'' = \{\mathcal{D}|_U \mid \mathcal{D} \in \mathcal{S}, U \in \mathcal{U}\} = \{\mathcal{D}|_U \mid \mathcal{D} \in \mathcal{S}', U \in \mathcal{U}\}$$

with \mathcal{U} being a sufficiently fine affine covering of Y . We then have

$$X(\mathcal{S}) \cong X(\mathcal{S}'') \cong X(\mathcal{S}').$$

For Y a complete curve we may also have polyhedral divisors with locus Y . For reconstructing $X(\mathcal{S})$ from the slices we need to know which polyhedra belong to divisors with complete loci.

In the forthcoming examples we will therefore label the maximal polyhedra in a subdivision by the polyhedral divisor they belong to. The locus of any polyhedral divisor $\mathcal{D} \in \mathcal{S}$ can then be read off immediately.

Remark 2.8. [AHS08, sec. 5] We get a very illuminating class of examples from toric geometry by restricting the torus action.

Let us consider a complete n -dimensional toric variety $X := X(\Sigma)$. We restrict its torus action to that of a smaller torus $T \hookrightarrow T_X$ and construct a divisorial fan \mathcal{S}

with $X(\mathcal{S}) = X(\Sigma)$ in the following way. The embedding $T \hookrightarrow T_X$ corresponds to an exact sequence of lattices

$$0 \rightarrow N \xrightarrow{F} N_X \xrightarrow{P'} N' \rightarrow 0.$$

We may choose a splitting $N_X \cong N \oplus N'$ with projections

$$P : N_X \rightarrow N, \quad P' : N_X \rightarrow N'.$$

Define $Y := X(\Sigma')$, where Σ' is an arbitrary smooth projective fan Σ' refining the images $P'(\delta)$ of all faces $\delta \in \Sigma$. Then every cone $\sigma \in \Sigma(n)$ gives rise to a polyhedral divisor \mathcal{D}_σ . For each ray $\rho' \in \Sigma'(1)$, let $n_{\rho'}$ denote its primitive generator. We then set

$$\Delta_{\rho'}(\sigma) = P(P'^{-1}(n_{\rho'}) \cap \sigma), \text{ and } \mathcal{D}_\sigma = \sum_{\rho' \in \Sigma'(1)} \Delta_{\rho'}(\sigma) \otimes D_{\rho'}.$$

Finally $\{\mathcal{D}_\sigma\}_{\sigma \in \Sigma(n)}$ is a divisorial fan. Observe that for certain polyhedral divisors \mathcal{D}_σ and rays $\rho' \in \Sigma'(1)$ the intersection $P'^{-1}(n_{\rho'}) \cap \sigma$ may be empty. In this case we have that $\Delta_{\rho'}(\sigma) = \emptyset$.

Example 2.9. We consider the Hirzebruch surface \mathbb{F}_a as a \mathbb{C}^* -surface via the following maps of lattices

$$F = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P' = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

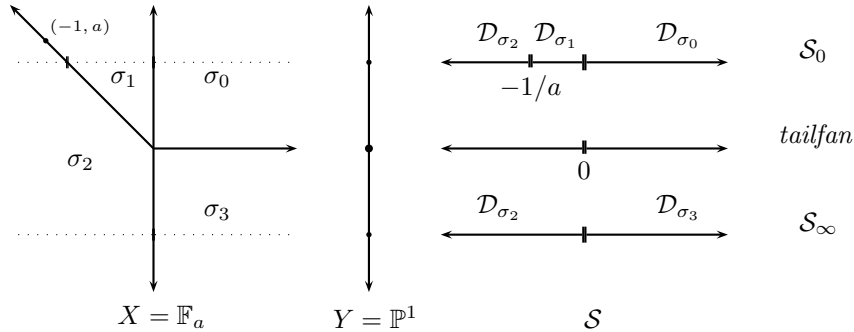


FIGURE 1. Divisorial fan associated to \mathbb{F}_a .

3. INVARIANT DIVISORS

As we saw in the previous section the case of a torus action of codimension one can be handled quite comfortably. In particular, this is true for the concept of divisors on T -varieties. Therefore, from now on we will restrict to this case unless stated otherwise.

Proposition 3.1. *Let \mathcal{D} be a polyhedral divisor with complete locus. Then $X := X(\mathcal{D})$ has a trivial invariant Picard group $\text{T-Pic}(X)$.*

Proof. Recall that the affine coordinate ring of $X(\mathcal{D})$ is

$$A := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))).$$

We choose an element $v \in \text{relint}(\text{tail } \mathcal{D})$ and consider the homogeneous prime ideal

$$A_{>0} := \bigoplus_{\langle u, v \rangle > 0} A_u.$$

As the locus of \mathcal{D} is complete we have $A_u = \Gamma(Y, \mathcal{O}_Y) = k$ for $\langle u, v \rangle = 0$. This implies that every nonzero homogeneous element in $A \setminus A_{>0}$ is a unit. Therefore, every homogeneous ideal is a subset of $A_{>0}$, i.e. $V := V(A_{>0})$ lies in the closure of every T -orbit on X . So the only invariant affine open subset that hits V is X itself. Hence, every invariant covering of X contains X itself implying that every invariant Cartier-divisor has to be principal. \square

3.1. Cartier divisors. Our aim is to give a description of an invariant Cartier divisor D on $X = X(\mathcal{S})$ in terms of a piecewise linear function on $\text{TV}(\Sigma)$ and a divisor on the curve Y , where Σ is the tailfan of \mathcal{S} . The crucial input is the notion of a *divisorial support function*.

Let $\Sigma \subset N_{\mathbb{Q}}$ be a complete polyhedral subdivision of $N_{\mathbb{Q}}$ consisting of tailed polyhedra.

Definition 3.2. A continuous function $h : |\Sigma| \rightarrow \mathbb{Q}$ which is affine on every polyhedron $\Delta \in \Sigma$ is called a \mathbb{Q} -*support function*, or merely a *support function* if it has integer slope and integer translation, viz. for $v \in |\Sigma|$ and $k \in \mathbb{N}$ such that kv is a lattice point we have $kh(v) \in \mathbb{Z}$. The group of support functions on Σ is denoted by $\text{SF}(\Sigma)$.

Definition 3.3. Let h be as above and $\Delta \in \Sigma$ a polyhedron with tail cone δ . We define a linear function h_t^Δ on δ by setting $h_t^\Delta(v) := h(p+v) - h(p)$ for some $p \in \Delta$. As h_t^Δ is induced by h we call it the *linear part* of $h|_\Delta$, or $\text{lin } h|_\Delta$ for short.

Using 3.2 and 3.3 we can obviously associate a unique continuous piecewise linear function with an element $h \in \text{SF}(\Sigma)$, say h_t . That is how we come to the crucial definition of this section.

Let \mathcal{S} be a divisorial fan on a curve Y . For every $P \in Y$ we thus get a polyhedral subdivision \mathcal{S}_P consisting of polyhedral coefficients.

Definition 3.4. We define $\text{SF}(\mathcal{S})$ to be the group of all collections

$$(h_P)_{P \in Y} \in \prod_{P \in Y} \text{SF}(\mathcal{S}_P) \quad \text{with}$$

- (1) all h_P have the same linear part h_t , i.e. for polytopes $\Delta \in \mathcal{S}_P$ and $\Delta' \in \mathcal{S}_{P'}$ with the same tailcone δ we have that $\text{lin } h_P|_\Delta = \text{lin } h_{P'}|_{\Delta'} = h_t|_\delta$.
- (2) only for finitely many $P \in Y$ h_P differs from h_t .

We call $\text{SF}(\mathcal{S})$ the group of *divisorial support functions* on \mathcal{S} .

Notation 3.5. We may restrict an element $h \in \text{SF}(\mathcal{S})$ to a subfan or even to a polyhedral divisor $\mathcal{D} \in \mathcal{S}$. The restriction will be denoted by $h|_{\mathcal{D}}$.

Definition 3.6. A divisorial support function $h \in \text{SF}(\mathcal{S})$ is called *principal* if $h(v) = \langle u, v \rangle + D$ with $u \in M$ and D is a principal divisor on Y . Here, D is to be considered as an element in $\text{SF}(\mathcal{S})$ taking the constant value $\text{coeff}_P(D)$ on every slice \mathcal{S}_P .

Remark 3.7. Let us denote the function field of Y by $K(Y)$. We then consider the graded ring $\bigoplus_{u \in M} K(Y)$, its multiplication being induced by the one in $K(Y)$. Hence, we have a canonical inclusion of graded rings

$$A := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(\mathcal{O}(\mathcal{D}(u))) \hookrightarrow \bigoplus_{u \in M} K(Y).$$

Moreover, the ring $\bigoplus_{u \in M} K(Y)$ is equal to the subring

$$K(X)^{\text{hom}} \subset K(X) = \text{Quot}(A)$$

which is generated by the semi-invariant functions on X , since X is birationally equivalent to $T_M \times Y$ as a consequence of [AH06, Thm. 3.1].

Thus, denoting the unit of $K(Y)$ in degree u by $\chi^u \in K(X)^{\text{hom}}$ we obtain a unique representation $f \cdot \chi^u$ with $f \in K(Y)$ and $u \in M$ for every semi-invariant function on X .

Definition 3.8. A divisorial support function h is called *Cartier* if for every $\mathcal{D} \in \mathcal{S}$ with complete locus its restriction $h|_{\mathcal{D}}$ is principal. The corresponding group is denoted by $\text{CaSF}(\mathcal{S})$.

Notation 3.9. The group of T -invariant Cartier divisors on $X(\mathcal{S})$ is denoted by $\text{T-CaDiv}(\mathcal{S})$.

Proposition 3.10. $\text{T-CaDiv}(\mathcal{S})$ and $\text{CaSF}(\mathcal{S})$ are isomorphic as abelian groups.

Proof. Take an element $h = (h_P)_P \in \text{CaSF}(\mathcal{S})$. Then there exist for every $\mathcal{D} \in \mathcal{S}$ a weight $u_P(\mathcal{D}) \in M$ and a constant $a_P(\mathcal{D}) \in \mathbb{Z}$ such that $h_P|_{\mathcal{D}}(v) = u_P(v) + a_P$. Observe that for achieving this we possibly have to pass to a refinement of \mathcal{S} as stated in 2.7. We may then cover Y by Y_i such that all $\sum a_P(\mathcal{D})P$ are principal on Y_i . Let us assume that $\sum a_P(\mathcal{D})P = -\text{div}(f_{\mathcal{D}}^i)$ on Y_i . Then $f_{\mathcal{D}}^i \cdot \chi^{u(\mathcal{D})} \in K(X)^T$, so it defines a principal divisor on $X(\mathcal{D})|_{Y_i}$. All of these principal divisors patch together to a Cartier divisor on the subvariety of $X(\mathcal{S})$ build up by the polyhedral divisors with affine locus.

Consider now an element $\mathcal{D} \in \mathcal{S}$ with complete locus. Then we know by 3.1 that every T -invariant Cartier divisor has to be principal on $X(\mathcal{D})$ and it is straightforward to see that principal support functions encode principal divisors.

On the other hand consider an element $D \in \text{T-CaDiv}(\mathcal{S})$ given by an invariant open affine covering $(U_i)_{i \in I}$ of $X(\mathcal{S})$ and elements $\tilde{f}_i \in K(X)^*$. Intersecting with the open covering coming from $(X(\mathcal{D}))_{\mathcal{D} \in \mathcal{S}}$ yields an open invariant cover for every affine $X(\mathcal{D})$ which itself is induced by a cover of $\text{Loc } \mathcal{D}$. One can easily construct h_t by picking for every maximal $\sigma \in \text{tailfan}$ an element \mathcal{D} having exactly this tailcone. Depending on whether $\text{Loc } \mathcal{D}$ is affine or complete we have

$$D|_{X(\mathcal{D})} = (U_j, f_j \chi^{u_j})_j \quad \text{or} \quad D|_{X(\mathcal{D})} = (X(\mathcal{D}), f \chi^u).$$

We assign $-u_j$ to σ . This choice is independent of j . For an element $\mathcal{D} \in \mathcal{S}$ with complete locus there is no choice since we only have the trivial covering. If $P \in Y$ is in $\text{supp}(\text{div } f_j)$ we set $a_P := \text{coeff}_P(\text{div } f_j)$. This again is independent of the choice of j . Furthermore these functions glue together to an element $h \in \text{CaSF}(\mathcal{S})$. \square

Notation 3.11. The T -invariant Cartier divisor induced by an element $h \in \text{CaSF}(\mathcal{S})$ is denoted by D_h . We will often switch from one notation to the other and consider both expressions to mean the same thing.

As an immediate consequence we have this

Corollary 3.12. *The T -invariant Picard group of $X(\mathcal{S})$ is given by*

$$\text{T-Pic } X \cong \frac{\text{CaSF}(\mathcal{S})}{\langle \langle u, \cdot \rangle + D \mid D \sim 0, u \in M \rangle}.$$

3.2. Weil divisors. We split this section into two parts as since we will give some results concerning torus actions of arbitrary codimension. We will deal with this case in the first part. The second part can be regarded as a specialization of the first one but we also provide further results not yet obtained in the general case.

Torus Actions of Arbitrary Codimension. We would like to describe T -invariant prime divisors. As $X(\mathcal{S})$ is patched together by affine charts $X(\mathcal{D})$ we can restrict to the affine case. Set $n := \dim(T) = \dim(X(\mathcal{S})) - k$, where k is the dimension of the base variety Y . We can assume the latter to be smooth and projective.

In general there are two types of T -invariant prime divisors:

- (1) families of n -dimensional orbit closures over prime divisors in $X(\mathcal{S})$.
- (2) families of $n - 1$ -dimensional orbit closures over $X(\mathcal{S})$.

Proposition 3.13. *Let \mathcal{D} be a polyhedral divisor on an arbitrary normal variety Y , then there are one-to-one correspondences*

- (1) *between prime divisor of type 1 and vertices $v \in \Delta_Z$ with Z being a prime divisor on Y , such that $\mathcal{O}(D(u))|_Z$ is big, for $u \in ((\Delta_Z - v)^\vee)^\circ$.*
- (2) *between prime divisors of type 2 and rays ρ of tail \mathcal{D} with $\mathcal{D}(u)$ big for $u \in (\rho^\vee)^\circ$.*

Proof. Consider $\tilde{X} := \mathbf{Spec}_{\mathrm{Loc} \mathcal{D}} \bigoplus_u \mathcal{O}(\mathcal{D}(u))$. We have $X(\mathcal{D}) = \mathbf{Spec} \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}})$ and get equivariant maps

$$\mathrm{Loc} \mathcal{D} \xleftarrow{\pi} \tilde{X} \xrightarrow{r} X.$$

From [AH06, 3.1] we know that π is a good quotient map, and r is a birational morphism. In [AH06, 7.11] the orbit structure of the fibers of π is described. Thus, we know that l -dimensional faces F of \mathcal{D}_y correspond to T -invariant closed subvarieties of codimension l in $\pi_y := \pi^{-1}(y)$. While stated only for closed points one checks that the proof in fact works equally well for generic points ξ .

Furthermore we have to consider those subvarieties that get contracted by r . By [AH06, 10.1] we have that

$$\dim Z - \dim r(Z) = \dim \pi(Z) - \dim \vartheta_u(\pi(Z))$$

for any invariant subvariety $Z \subset \tilde{X}$.

So, the bigness condition is equivalent to the fact that the image under r of the corresponding prime divisor in \tilde{X} is again of codimension 1. \square

Proposition 3.14. *We consider a polyhedral divisor on an arbitrary normal variety Y . Let $f \cdot \chi^u \in K(X)^{\mathrm{hom}}$. Then the corresponding principal divisor is given by*

$$-\sum_{\rho} \langle u, n_{\rho} \rangle D_{\rho} - \sum_{(Z,v)} \mu(v) (\langle u, v \rangle + \mathrm{ord}_Z f) D_{(Z,v)},$$

where $\mu(v)$ is the smallest integer $k \geq 1$ such that $k \cdot v$ is a lattice point, this lattice point is a multiple of the primitive lattice vector: $\mu(v)v = \varepsilon(v)n_v$.

Proof. This is a local statement, so we will pass to a sufficiently small invariant open affine set which meets a particular prime divisor. If we translate this to into our combinatorial language and consider a prime divisor corresponding to (Z, v) or ρ then we have to choose a polyhedral divisor $\mathcal{D}' \prec \mathcal{D} \in \mathcal{S}$ such that v is also a vertex of \mathcal{D}'_Z or ρ is a ray in tail \mathcal{D}' , respectively.

So we restrict to following two affine cases:

- (1) \mathcal{D} is a polyhedral divisor with tail cone $\sigma = 0$ and a single point $\Delta_Z = \{v\} \subset N$ as the only nontrivial coefficient. Moreover, Y is affine and factorial. In particular, Z is a prime divisor with (local) parameter t_Z .
- (2) \mathcal{D} is the trivial polyhedral divisor with one dimensional tail cone ρ over an affine locus Y .

In the first case we may choose a \mathbb{Z} -Basis e_1, \dots, e_m of N with $e_1 = n_v$ and consider the dual basis e_1^*, \dots, e_m^* . By definition $\varepsilon(v)$ and $\mu(v)$ are coprime, so we

will find $a, b \in \mathbb{Z}$ such that $a\mu(v) + b\varepsilon(v) = 1$. In this situation $y := t_Z^a \chi^{be_1^*}$ is irreducible in

$$\Gamma(\mathcal{O}_X) = \Gamma(\mathcal{O}_Y)[y, t_Z^{\pm\varepsilon(v)} \chi^{\mp\mu(v)e_1^*}, \chi^{\pm e_2^*}, \dots, \chi^{\pm e_m^*}],$$

and defines the prime divisor $D_{(Z,v)}$. We consider an element $t_Z^\alpha \chi^u$ with $u = \sum_i \lambda_i e_i^*$. The y -order of $t_Z^\alpha \chi^u$ is

$$\varepsilon(v)\lambda_1 + \mu(v)\alpha = \mu(v)(\langle u, v \rangle + \alpha),$$

since $t_Z^\alpha \chi^u = y^{\varepsilon(v)\lambda_1 + \mu(v)\alpha} (t_Z^{-\varepsilon(v)} \chi^{\mu(v)e_1^*})^{\lambda_1 a + b\alpha}$, and $t_Z^{-\varepsilon(v)} \chi^{\mu(v)e_1^*}$ is a unit.

In the second case we choose a \mathbb{Z} -basis e_1, \dots, e_m of N with $e_1 = n_\rho$. Once again we consider the dual basis e_1^*, \dots, e_m^* . In this situation

$$\Gamma(\mathcal{O}_X) = \Gamma(\mathcal{O}_Y)[\chi^{e_1^*}, \chi^{\pm e_2^*}, \dots, \chi^{\pm e_m^*}].$$

Now $(\chi^{e_1^*})$ defines the prime divisor ρ on X . For a principal divisor $f \cdot \chi^u$, the $\chi^{e_1^*}$ -order equals the e_1^* -component of u , i.e. $\langle u, n_\rho \rangle$. \square

Our next goal is to describe the divisor class group of $X(\mathcal{S})$. Denote by $\text{T-Div}(X(\mathcal{S}))$ the T -invariant divisors, and by $\text{T-Prin}(X(\mathcal{S}))$ the T -invariant principal divisors on $X(\mathcal{S})$. Then we have that

$$\text{Cl } X(\mathcal{S}) \cong \text{T-Cl}(X(\mathcal{S})) := \frac{\text{T-Div}(X(\mathcal{S}))}{\text{T-Prin}(X(\mathcal{S}))}.$$

Corollary 3.15. *The divisor class group of $X(\mathcal{S})$ is given by*

$$\text{Cl } X(\mathcal{S}) = \frac{\bigoplus_\rho \mathbb{Z} \cdot D_\rho \oplus \bigoplus_{D_{(Z,v)}} \mathbb{Z} \cdot D_{(Z,v)}}{\langle \sum u(n_\rho) D_\rho + \sum_{D_{(Z,v)}} \mu(v)(\langle u, v \rangle + a_Z) D_{(Z,v)} \rangle}.$$

Here u runs over all elements of M and $\sum_Z a_Z Z$ over all principal divisors on Y . Thus it is isomorphic to

$$\text{Cl } Y \oplus \bigoplus_\rho \mathbb{Z} D_\rho \oplus \bigoplus_{D_{(Z,v)}} \mathbb{Z} D_{(Z,v)}$$

modulo the relations

$$\begin{aligned} [Z] &= \sum_{v \in \mathcal{S}_Z} \mu(v) D_{(Z,v)}, \\ 0 &= \sum_\rho \langle u, \rho \rangle D_\rho + \sum_{D_{(Z,v)}} \mu(v) \langle u, v \rangle D_{(Z,v)}. \end{aligned}$$

Remark 3.16. We can also describe the ideals of prime divisors in terms of polyhedral divisors:

- (1) For prime divisors of type 1 corresponding to (Z, v) the ideal is given by

$$I_{D_{(Z,v)}} = \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) \cap \{f \in K(Y) \mid \text{ord}_Z(f) > \langle u, v \rangle\}.$$

- (2) For prime divisors of type 2 the corresponding ideal is generated by all degrees u which are not orthogonal to ρ :

$$I_{D_\rho} = \bigoplus_{u \in \sigma^\vee \setminus \rho^\perp \cap M} \Gamma(Y, \mathcal{O}(\mathcal{D}(u))).$$

Torus Actions of Codimension One. Stepping back to the codimension one case 3.13 is equivalent to

Corollary 3.17. *Let \mathcal{D} be a polyhedral divisor on a curve Y . Then there are one-to-one correspondences*

- (1) *between prime divisors of type 1 and pairs (P, v) with P being point on Y and v a vertex of Δ_P ,*
- (2) *between prime divisors of type 2 and rays ρ of σ with $\deg \mathcal{D} \cap \rho = \emptyset$.*

Definition 3.18. Let $\mathcal{D} \in \mathcal{S}$ be a polyhedral divisor with tailcone σ . A ray $\rho \prec \sigma$ with $\deg \mathcal{D} \cap \rho = \emptyset$ is called an *extremal ray*. The set of extremal rays is denoted by $\text{x-rays}(\mathcal{D})$ or $\text{x-rays}(\mathcal{S})$, respectively.

The combination of 3.14 and the description of T -Cartier divisors yield

Corollary 3.19. *Let $h = \sum_P h_P$ be a Cartier divisor on \mathcal{D} . Then the corresponding Weil divisor is given by*

$$-\sum_{\rho} h_t(n_{\rho}) D_{\rho} - \sum_{(P,v)} \mu(v) h_P(v) D_{(P,v)}.$$

Corollary 3.20. *Assume $X = X(\mathcal{S})$ to be a complete \mathbb{Q} -factorial variety of dimension $n + 1$. Denote by $\mathcal{S}_P^{(0)}$ the set of vertices in S_P . Then the Picard number of X is given by*

$$\rho_X = 1 + \# \text{x-rays}(\mathcal{S}) + \sum_{P \in Y} (\# \mathcal{S}_P^{(0)} - 1) - n.$$

Theorem 3.21. *For the canonical class of $X = X(\mathcal{S})$ we have*

$$K_X = \sum_{(P,v)} (\mu(v) K_Y(P) + \mu(v) - 1) \cdot D_{(P,v)} - \sum_{\rho} D_{\rho}.$$

Proof. Let $\omega_Y \in \Omega^1(Y)$ a (rational) differential form. Then K_Y is given by $K_Y = \text{div } \omega_Y$. For a given $P \in Y$ we have a representation $\omega_Y = f_P dt_P$, where $f_P \in K(Y)$ and t_P a local parameter of P

We define a differential form ω_X by

$$\omega_X = \omega_Y \wedge \frac{d\chi_1^{e_1^*}}{\chi_1^{e_1^*}} \wedge \dots \wedge \frac{d\chi_n^{e_n^*}}{\chi_n^{e_n^*}},$$

with e_1^*, \dots, e_n^* being a \mathbb{Z} -basis of M .

For a prime divisor (P, v) we may choose a \mathbb{Z} -Basis e_1, \dots, e_n of N with $e_1 = n_v$. Consider the dual basis e_1^*, \dots, e_n^* . As μ and $\varepsilon(v)$ are coprime we may choose $a, b \in \mathbb{Z}$ with $a\mu(v) + b\varepsilon(v) = 1$. Hence $t_P^a \chi^{be_1^*}$ is a local parameter associated to (P, v) . It is then easy to see that we have the following local representation

$$\omega_X = \frac{f_P}{at_P^{a-1} \chi^{be_1^*}} d(t_P^a \chi^{be_1^*}) \wedge \frac{d\chi_1^{e_1^*}}{\chi_1^{e_1^*}} \wedge \dots \wedge \frac{d\chi_n^{e_n^*}}{\chi_n^{e_n^*}}.$$

Then 3.19 implies $\text{ord}_{D_{(P,v)}}(\frac{f_P}{at_P^{a-1} \chi^{be_1^*}}) = (\text{ord}_P(f_P) + 1)\mu(v) - 1$.

For a prime divisor D_{ρ} of X we choose a \mathbb{Z} -basis e_1, \dots, e_n of N with $e_1 = n_{\rho}$. Again we consider the dual basis e_1^*, \dots, e_n^* . Then $\chi^{e_1^*}$ is a local parameter for D_{ρ} and we have a local representation

$$\omega_X = \omega_Y \wedge \frac{d\chi_1^{e_1^*}}{\chi_1^{e_1^*}} \wedge \dots \wedge \frac{d\chi_n^{e_n^*}}{\chi_n^{e_n^*}},$$

We immediately find that $\text{ord}_{D_{\rho}}(\omega_Y) = 0$ and $\text{ord}_{D_{\rho}}(\frac{d\chi_1^{e_1^*}}{\chi_1^{e_1^*}} \wedge \dots \wedge \frac{d\chi_n^{e_n^*}}{\chi_n^{e_n^*}}) = -1$. \square

3.3. Global sections. For an invariant Cartier-Divisor D_h on X we may consider the M -graded module of global sections

$$L(h) = \bigoplus_{u \in M} L(h)_u = \bigoplus_{u \in M} L(D_h)_u := \Gamma(X, \mathcal{O}(D_h)).$$

The weight set of h is defined as $\{u \in M \mid L(h)_u \neq 0\}$. For a Cartier divisor D_h given by $h \in \text{CaSF}(\mathcal{S})$ we will bound its weight set by a polyhedron and describe the graded module structure of $L(h)$.

Definition 3.22. Given a support function $h = (h_P)_P$ with linear part h_t its associated polytope is given by

$$\square_h := \square_{h_t} := \{u \in M \mid \langle u, v \rangle \geq h_t(v) \ \forall v \in N\}.$$

We furthermore define the dual function $h^* : \square_h \rightarrow \text{Div}_{\mathbb{Q}} Y$ to h by

$$h^*(u) := \sum_P h_P^*(u) P := \sum_P \min_{\text{vert}}(u - h_P) P,$$

where $\min_{\text{vert}}(u - h_P)$ denotes the minimal value of $u - h_P$ on the vertices of \mathcal{S}_P .

Proposition 3.23. Let $h \in \text{T-CaDiv}(\mathcal{S})$ be a Cartier divisor with linear part h_t . Then

- (1) The weight set of $L(h)$ is a subset of \square_h .
- (2) For $u \in \square_h$ we have

$$L(h)_u = \Gamma(Y, \mathcal{O}(h^*(u))).$$

Proof. By definition of $\mathcal{O}(D_h)$ we have

$$\Gamma(X, \mathcal{O}(h))^T = \{\chi^u f \mid \text{div}(\chi^u f) - \sum_{\rho} h_t(n_{\rho}) D_{\rho} - \sum_{(P,v)} \mu(v) h_P(v) D_{(P,v)} \geq 0\}.$$

But $\text{div}(\chi^u f) = \sum_{\rho} \langle u, n_{\rho} \rangle D_{\rho} + \sum_{(P,v)} \mu(v) (\langle u, v \rangle + \text{ord}_P(f)) D_{(P,v)}$, so for $\chi^u f \in L(h)$ we get the following bounds:

- (1) $\langle u, n_{\rho} \rangle \geq h_t(n_{\rho}) \ \forall \rho$
- (2) $\text{ord}_P(f) + \langle u, v \rangle \geq h_P(v) \ \forall (P, v)$

The first implies, that $u \in \square_{h_t}$, the second that $\text{ord}_P(f) + (u - h_P)(v) \geq 0 \ \forall (P, v)$. \square

Example 3.24. Let us consider the Hirzebruch surface \mathbb{F}_2 together with the line bundle $L = \mathcal{O}(D_{h_{\text{tor}}})$ which is given through the following generators on each cone σ_i

$$u_{\sigma_0} = [0 \ 0], \quad u_{\sigma_1} = [1 \ 0], \quad u_{\sigma_2} = [3 \ 1], \quad u_{\sigma_3} = [0 \ 1].$$

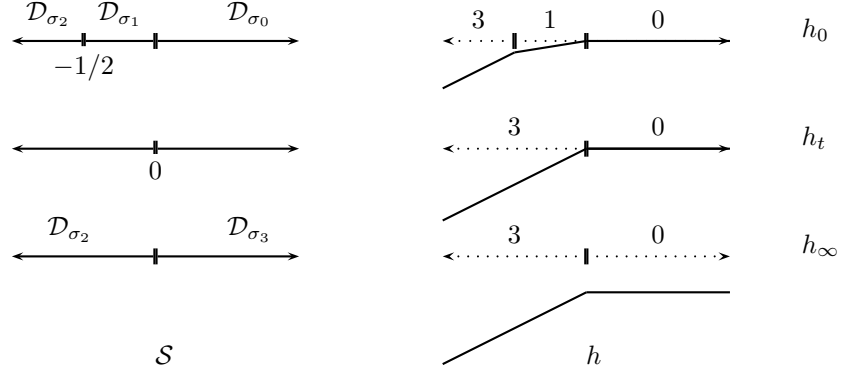
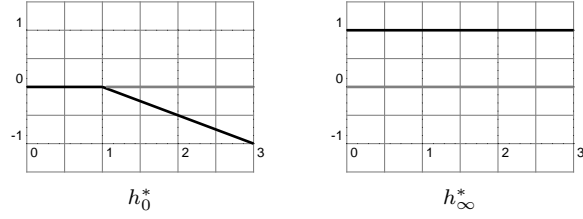
It is very ample and defines an embedding into \mathbb{P}^5 . One can describe the embedding by a polytope $\Delta \subset M_{\mathbb{Q}} = \mathbb{Q}^2$ which is the convex hull of the u_{σ_i} and has six lattice points which are a basis of $\Gamma(\mathbb{F}_2, \mathcal{O}(D_h))$.

Considering the piecewise linear function h_{tor} associated to L , one gains the graph of h_y over \mathcal{S}_y by evaluating h_{tor} along the dotted slices corresponding to y . As usual h_t denotes the piecewise linear function on the tailfan. The number over each cone denotes the slope of the corresponding restriction of h_y .

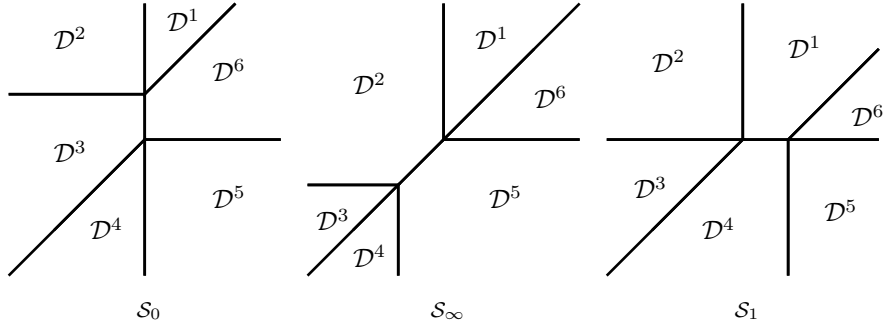
By 3.23 we have $\square_h = \{u \in \mathbb{Z} \mid 3 \geq u \geq 0\}$, and

$$\begin{aligned} L(h)_0 &= \Gamma(\mathbb{P}^1, \mathcal{O}(\{\infty\})), & L(h)_1 &= \Gamma(\mathbb{P}^1, \mathcal{O}(\{\infty\})), \\ L(h)_2 &= \Gamma(\mathbb{P}^1, \mathcal{O}(\{\infty\} - 1/2\{0\})), & L(h)_3 &= \Gamma(\mathbb{P}^1, \mathcal{O}(\{\infty\} - \{0\})). \end{aligned}$$

Altogether they sum up to a six dimensional vector space. We complete the example by a figure of h^* .

FIGURE 2. The graph of h_y over the corresponding slice \mathcal{S}_y .FIGURE 3. The graph of h_y^* over \square_h .

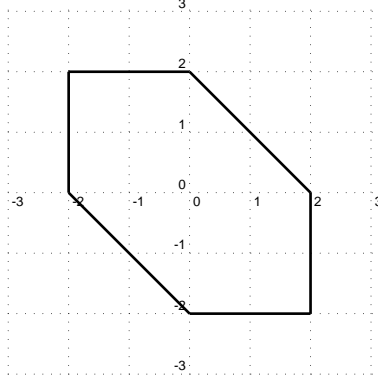
Example 3.25. As another example consider $X = \mathbb{P}(\Omega_{\mathbb{P}^2})$ which is a complete threefold X with a two dimensional torus action. Its divisorial fan \mathcal{S} over \mathbb{P}^1 looks like figure 4, cf. [AHS08, 8.5].

FIGURE 4. Divisorial fan of $\mathbb{P}(\Omega_{\mathbb{P}^2})$.

Note that all polyhedral divisors have complete locus. We want to compute $\Gamma(X, -K_X)$, and use $K_{\mathbb{P}^1} = -2\{0\}$ as a representation of the canonical divisor on \mathbb{P}^1 . By 3.21

$$-K_X = 2D_{(\{0\}, (0,0))} + 2D_{(\{0\}, (0,1))}.$$

Using 3.19 we can construct h explicitly. We have $h_t(\rho_i) = -2$ for $1 \leq i \leq 6$, providing the weight polytope \square_h in figure 5.

FIGURE 5. The weight polytope \square_h .

The next list displays the induced divisor $h^*(u)$ on \mathbb{P}^1 for every weight $u = (u_1, u_2) \in \square_h$, where a triple (a, b, c) corresponds to $D(a, b, c) = a\{0\} + b\{\infty\} + c\{1\}$.

$$\begin{array}{llll}
(0, 0) & - & (2, 0, 0) & (0, -1) & - & (1, 0, 0) & (1, 1) & - & (2, -2, 0) \\
(1, 0) & - & (2, -1, 0) & (0, -2) & - & (0, 0, 0) & (1, -1) & - & (1, 0, 0) \\
(2, 0) & - & (2, -2, 0) & (-1, 1) & - & (2, 0, -1) & (2, -1) & - & (1, -1, 0) \\
(-1, 0) & - & (2, 0, -1) & (-2, 1) & - & (2, 0, -2) & (2, -2) & - & (0, 0, 0) \\
(-2, 0) & - & (2, 0, -2) & (-2, 2) & - & (2, 0, -2) & (1, -2) & - & (0, 0, 0) \\
(0, 1) & - & (2, -1, 0) & (-1, 2) & - & (2, -1, -1) & & & \\
(0, 2) & - & (2, -2, 0) & (-1, -1) & - & (1, 0, -1) & & &
\end{array}$$

Summing up yields $\dim \Gamma(X, -K_X) = 27$. Furthermore we compute

$$\rho_X = 1 + 0 + 3 - 2 = 2,$$

which is of course a classical result.

3.4. Positivity of line bundles. The goal of this section is to determine criteria for the ampleness of an invariant Cartier divisor and to give a method how to compute intersection numbers of semiample invariant Cartier divisors. We assume \mathcal{S} to be complete. Denote its tailfan by Σ .

Definition 3.26. For a cone $\sigma \in \Sigma(n)$ of maximal dimension in the tail fan and a point $P \in Y$ we get exactly one polyhedron $\Delta_P^\sigma \in \mathcal{S}_P$ having tail σ . For a given support function $h = (h_P)_P$ we have

$$h_P|_{\Delta_P^\sigma} = \langle u^h(\sigma), \cdot \rangle + a_P^h(\sigma).$$

The constant part gives rise to a divisor on Y :

$$h|_\sigma(0) := \sum_P a_P^h(\sigma)P.$$

Theorem 3.27. A T -Cartier divisor $h \in \text{T-CaDiv}(\mathcal{S})$ is semiample iff all h_P are concave and $\deg h|_\sigma(0) < 0$ or some multiple of $-h|_\sigma(0)$ is principal, i.e. $-h|_\sigma(0)$ is semiample.

Proof. We first show that semiampleness follows from the above criteria. If h is concave then so is h_0 . This implies that the $u^h(\sigma)$ are exactly the vertices of \square_h and $h^*(u^h(\sigma)) = h|_\sigma(0)$. The semiampleness for $h^*(u)$, $u \in \square_h \cap M$ now follows from the semiampleness at the vertices. Indeed, if D, D' are semiample divisors on Y then $D + \lambda(D' - D)$ with $0 \leq \lambda \leq 1$ is also semiample. Observe that every vertex (u, a_u) of the graph $\Gamma_{h_P}^*$ corresponds to an affine piece of h_P . This again corresponds to a

function $f\chi^u$ with $\operatorname{div}(f) = a_u P$ on U_P for some $\mathcal{D} \in \mathcal{S}$ (see 3.1) with $P \in U_P \subset Y$ a sufficiently small neighborhood. Now $D_h|_{X(\mathcal{D}|_{U_P})} = \operatorname{div}(f^{-1}\chi^{-u})$.

A point $(u, a_u) \in M \times \mathbb{Z}$ is a vertex of the graph Γ_{h^*} iff (mu, ma_u) is a vertex of the graph $\Gamma_{(m \cdot h)^*}$. Hence, after passing to a suitable multiple of h we may assume that $h^*(u)$ is basepoint free with f being a global section of $\mathcal{O}(h^*(u))$. Then $f\chi^u$ is a global section of $\mathcal{O}(D_h)$ generating $\mathcal{O}(D_h)|_{X(\mathcal{D}|_{U_P})}$.

For the other implication assume that there is a point $P \in Y$ such that h_P is not concave. Then the same is true for all multiples lh_P . So we can find an affine part $\langle u, \cdot \rangle - a_u$ of lh_P such that $a_u > (lh_P)^*(u)$. But this implies that there is no global section $f\chi^u$ such that $\operatorname{div}(f) = a_u$ which contradicts the basepoint freeness of D_{lh} . Hence D_h cannot be semiample. \square

Corollary 3.28. *A T -Cartier divisor $h \in \operatorname{T-CaDiv}(\mathcal{S})$ is ample iff all h_P are strictly concave and for all tail cones σ belonging to a polyhedral divisor $\mathcal{D} \in \mathcal{S}$ with affine locus $\deg h|_\sigma(0) = \sum_P a_P^h(\sigma) < 0$, i.e. $-h|_\sigma(0)$ is ample.*

Proof. Note that for every invariant Cartier divisor D_h the concaveness of h implies that $h_\sigma(0)$ is principal. Hence, the proof follows from 3.27 and the fact that h_P is strictly concave if and only if for every support function h' there is a $k \gg 0$ such that $h' + kh_P$ is concave. \square

Corollary 3.29. *A T -Cartier divisor $h \in \operatorname{T-CaDiv}(\mathcal{S})$ is nef iff all h_P are concave and $\deg h|_\sigma \leq 0$ for every maximal cone $\sigma \in \Sigma(n)$.*

Proof. Using the equivariant Chow Lemma we can pull back D_h by an equivariant birational proper morphism $\phi : X(\mathcal{S}') \rightarrow X(\mathcal{S})$. So we may assume that D_h lives on a projective T -variety $X' := X(\mathcal{S}')$, i.e. there exists an ample divisor $D_{h'}$ on X' . It is easy to check that $D_h + \varepsilon D_{h'}$ is ample for $\varepsilon > 0$ iff h fulfills the above conditions. \square

Using proposition 3.23 to determine $\dim \Gamma(X, D_h)$ we are now able to compute intersection numbers.

Definition 3.30. For a function $h^* : \square \rightarrow \operatorname{Div}_{\mathbb{Q}} Y$ we define its *volume* to be

$$\operatorname{vol} h^* := \sum_P \int_{\square} h_P^* \operatorname{vol}_{M_{\mathbb{R}}}.$$

We associate a *mixed volume* to functions h_1^*, \dots, h_k^* by setting

$$V(h_1^*, \dots, h_k^*) := \sum_{i=1}^k (-1)^{i-1} \sum_{1 \leq j_1 \leq \dots \leq j_i \leq k} \operatorname{vol}(h_{j_1}^* + \dots + h_{j_i}^*).$$

Proposition 3.31. *Let \mathcal{S} be a divisorial fan on a curve Y with slices in $N \cong \mathbb{Z}^n$.*

(1) *The self-intersection number of a semiample Cartier divisor D_h is given by*

$$(D_h)^{(n+1)} = (n+1)! \operatorname{vol} h^*.$$

(2) *Assume that h_1, \dots, h_{n+1} define semiample divisors D_i on $X(\mathcal{S})$. Then*

$$(D_1 \cdots D_{n+1}) = (n+1)! V(h_1^*, \dots, h_{n+1}^*).$$

Proof. If we apply (1) to every sum of divisors from D_1, \dots, D_{m+1} we get (2) by the multilinearity and the symmetry of intersection numbers. To prove (1) we first recall that

$$(D_h)^{m+1} = \lim_{\nu \rightarrow \infty} \frac{(m+1)!}{\nu^{m+1}} \chi(X, \mathcal{O}(\nu D_h)).$$

Invoking the equivariant Chow Lemma we can assume $X := X(\mathcal{S})$ to be projective. So the higher cohomology groups are asymptotically irrelevant [Dem01, Thm. 6.7.]. Hence

$$(D_h)^{m+1} = \lim_{\nu \rightarrow \infty} \frac{(m+1)!}{\nu^{m+1}} h^0(X, \mathcal{O}(\nu D_h)).$$

Note that $(\nu h)^*(u) = \nu \cdot h^*(\frac{1}{\nu}u)$. We can now bound h^0 by

$$\begin{aligned} \sum_{u \in \nu \square_h \cap M} (\deg \lfloor \nu h^*(\frac{1}{\nu}u) \rfloor - g(Y) + 1) &\leq h^0(\mathcal{O}(\nu D_h)) \\ (a) \quad &\leq \sum_{u \in \nu \square_h \cap M} \deg \lfloor \nu h^*(\frac{1}{\nu}u) \rfloor + 1. \end{aligned}$$

Furthermore we have

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \frac{(m+1)!}{\nu^{m+1}} \sum_{u \in \nu \square_h \cap M} \deg \lfloor \nu h^*(\frac{1}{\nu}u) \rfloor &= \lim_{\nu \rightarrow \infty} \frac{(m+1)!}{\nu^m} \sum_{u \in \square_h \cap \frac{1}{\nu}M} \frac{1}{\nu} \deg \lfloor \nu h^*(u) \rfloor \\ &= (m+1)! \int_{\square_h} h^* \text{vol}_{M_{\mathbb{R}}} . \end{aligned}$$

Finally,

$$\lim_{\nu \rightarrow \infty} \frac{1}{\nu^{m+1}} \sum_{u \in \nu \square_h \cap M} (g-1) = (g-1) \lim_{\nu \rightarrow \infty} \frac{\#(\nu \cdot \square_h \cap M)}{\nu^{m+1}} = 0 .$$

Passing to the limit in (a), the term in the middle converges to $\text{vol } h^*$. This completes the proof. \square

Example 3.32. Let $X = \mathbb{P}_{\Omega_{\mathbb{P}^2}}$ as in 3.25 and $D_h = -K_X$. An easy calculation now shows that $(-K_X)^3 = 6 \cdot (18\frac{2}{3} - 5\frac{1}{3} - 5\frac{1}{3}) = 48$, matching the result already known from the classification of Fano threefolds.

4. COMPARING RESULTS IN THE CASE OF AFFINE \mathbb{C}^* -SURFACES

Normal affine \mathbb{C}^* -surfaces are very well understood. Results concerning the divisor class group, and the canonical divisor can be found in [FZ03, 4.24, 4.25] and references therein. We will shortly remind the reader of the notation used in [FZ03] where the Dolgachev-Pinkham-Demazure (DPD) construction is used for the explicit construction of (hyperbolic) affine \mathbb{C}^* -surfaces, and state the corresponding results.

4.1. Elliptic Case. Let C be a smooth projective curve and $D = \sum_i \frac{e_i}{m_i} [a_i]$ a \mathbb{Q} -Cartier divisor with $\sum_i \frac{e_i}{m_i} > 0$ and $\gcd(e_i, m_i) = 1$. The cone construction provides an affine \mathbb{C}^* -surface $X = \text{Spec } A_{C,D}$ whose class group of divisors is given by

$$\text{Cl } X = \frac{\text{Pic } C \oplus \bigoplus_i \mathbb{Z}[O_i]}{\langle \pi^*(a_i) = m_i[O_i], 0 = \sum_i e_i[O_i] \rangle},$$

and the canonical divisor can be represented by

$$K_X = \pi^* K_C = \sum_i (m_i - 1)[O_i].$$

Here, $O_i = \pi^{-1}(a_i)$. The corresponding data in our language are $Y = C$, $\sigma = \mathbb{Q}_{\geq 0}$, and $\mathcal{D} = \sum_i [\frac{e_i}{m_i}, \infty) \otimes a_i$. Then $D = \mathcal{D}(1)$.

4.2. Parabolic Case. This time one considers a smooth affine curve C together with a \mathbb{Q} -divisor $D = \sum_i \frac{e_i}{m_i} [a_i]$ with $\gcd(e_i, m_i) = 1$. The DPD construction yields an affine surface $X = \text{Spec } A_{C,D}$ whose class group of divisors has the following form

$$\text{Cl } X = \frac{\text{Pic } C \oplus \mathbb{Z}[C] \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i]}{\langle \pi^*(a_i) = m_i[O_i], [C] = -\sum_{i=1}^k e_i[O_i] \rangle}.$$

In addition one has that

$$K_X = \pi^* K_C + \sum_{i=1}^k (m_i - 1)[O_i] - [C].$$

Again, $O_i = \pi^{-1}(a_i)$. Using our notation gives $Y = C$, $\sigma = \mathbb{Q}_{\geq 0}$, and $\mathcal{D} = \sum_i [\frac{e_i}{m_i}, \infty) \otimes a_i$. Once more $D = \mathcal{D}(1)$.

4.3. Hyperbolic Case. Let us consider a smooth affine curve C and a pair (D_+, D_-)

$$D_+ = -\sum_i \frac{e_i}{m_i} a_i - \sum_j \frac{e_j^+}{m_j^+} b_j, \quad D_- = \sum_i \frac{e_i}{m_i} a_i + \sum_j \frac{e_j^-}{m_j^-} b_j$$

of \mathbb{Q} -divisors on C such that $D_+ + D_- \leq 0$. Recall the convention that

$$D_+(a_i) + D_-(a_i) = 0, \quad \text{and} \quad D_+(b_j) + D_-(b_j) < 0.$$

Using this pair the DPD construction provides us with an affine \mathbb{C}^* -surface

$$X = \text{Spec } A_{C,(D_+,D_-)}.$$

The class group of divisors $\text{Cl } X$ then is

$$\text{Pic } C \oplus \sum_{i=1}^k \mathbb{Z}[O_i] \oplus \sum_{j=1}^l (\mathbb{Z}[\overline{O}_j^+] \oplus \mathbb{Z}[\overline{O}_j^-])$$

modulo the relations

$$\begin{aligned} \pi^*(a_i) &= m_i[O_i] \\ \pi^*(b_j) &= m_j^+[\overline{O}_j^+] - m_j^-[\overline{O}_j^-] \\ 0 &= \sum_{i=1}^k e_i[O_i] + \sum_{j=1}^l (e_j^+[\overline{O}_j^+] - e_j^-[\overline{O}_j^-]). \end{aligned}$$

Furthermore,

$$K_X = \pi^* K_C + \sum_{i=1}^k (m_i - 1)[O_i] + \sum_{j=1}^l ((m_j^+ - 1)[\overline{O}_j^+] + (-m_j^- - 1)[\overline{O}_j^-]).$$

As before, $O_i = \pi^{-1}(a_i)$, whereas $\pi^{-1}(b_j) = \overline{O}_j^+ \cup \overline{O}_j^-$. In our terms: $Y = C$, $\sigma = \{0\}$ and $\mathcal{D} = \sum_i [v_i^-, v_i^+] \otimes y_i$ with $D_+ = \mathcal{D}(1)$, and $D_- = \mathcal{D}(-1)$.

Using our formulae for the divisor class group and the canonical divisor yields the same results in all three cases, as can readily be seen by invoking 3.15 and 3.21. So treating each case separately is no longer necessary. Another advantage is that these formulae can easily be applied to higher dimensional varieties with codimension one torus action. Furthermore, the language we use seems to yield a more natural means to study and treat the case of a complete T -variety, as to some extent has already been pointed out in [AHS08, 8.1].

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